Lecture 3 (Basics of set theory)

Set: A set is defined as a well defined collection of well defined distinct objects. The objects are called the elements or members of the set. We denote a set usually by capital letters, such as, $A, B, X, Y, \ldots$, whereas the lower-case letters $a, b, p, q, \ldots$ will usually be used to denote elements of sets. The set having no element is called empty set or nul set denoted by ϕ. If x is an element of a set X then we denote it by $x \in X$. The cardinality or the number of elements in a set S is denoted by $|S|$.

Let A and B be two sets such that elements of A are also the elements of B then we say that A is a subset of B, denoted as $A \subseteq B$. Two sets A and B are said to be equal, written as $A = B$, if $A \subseteq B$ and $B \subseteq A$. Let A be a set. A collection of all subsets of A is called power set of A, denoted as $P(A)$. If $|A| = n$, then $|P(A)| = 2^n$.

Examples:

1. Natural numbers $\mathbb{N}$, Integers $\mathbb{Z}$, Rationals $\mathbb{Q}$, Real numbers $\mathbb{R}$.
2. The solution of the equation $x^2 - 4x + 4$.
3. The set of nobel laureates in the world.
4. The set of points in $\mathbb{R}^2$.
5. The people living in India.

Operations on sets Let U be the universal set and A and B be two of its subsets. Then:

1. A union B, denoted as: $A \cup B = \{x \,|\, x \in A \vee x \in B\}$.
2. A intersection B, denoted as: $A \cap B = \{x \,|\, x \in A \wedge x \in B\}$.
3. A minus B denoted as: $A - B = \{x \,|\, x \in A \wedge x \notin B\}$.
4. A complement, denoted as: $A^c = \{x \,|\, x \notin A\}$, where U is universal set.
5. The symmetric difference of A and B written as: $A \oplus B = (A \cup B) - (A \cap B)$.

Set Identities Let A and B be two sets and U be universal set. Then:

1. Identity Laws: $A \cup \varnothing = A$ and $A \cap U = A$
2. Associative Laws: $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$
3. Commutative Laws: $A \cup B = B \cup A$ and $A \cap B = B \cap A$
4. Distributive Laws: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
5. De Morgan's Laws: $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$.
6. Complementation Law: $(A^c)^c = A$

Inclusion and exclusion principle:

1. For two sets A_1 and A_2: $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$.

2. For three sets A_1, A_2 and A_3: $|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$.
3. General form: $|\bigcup\limits_{i=1}^{n} A_i| = \sum_{i=1}^{n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| + \ldots + (-1)^{n-1} |A_1 \cap A_2 \cap \ldots \cap A_n|$.

Cartesian product: Let A and B be two sets. Then the cartesian product $A \times B$ of the sets is defined as $A \times B = \{(a, b) : a \in A, b \in B\}$.

- The elements of $A \times B$ are called ordered pairs.
- If $|A| = n$, $|B| = m$, then $|A \times B| = n \times m$.
- The Cartesian product $A \times B$ and $B \times A$ are not equal, unless $A = \varnothing$ or $B = \varnothing$.
- The Cartesian product of the sets $A_1, A_2, \ldots, A_n$, denoted by $A_1 \times A_2 \times \ldots \times A_n$, is the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$, where $a_i \in A_i$ for $i = 1, 2, \ldots, n$.

Examples 1: Let $A = \{1, 2\}$ and $B = \{a, b, c\}$. Then $A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$, $B \times A = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}$, and $A \times A = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$.

Example 2: Let $A = \mathbb{R}$. Then $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$.

Family of sets: Let I be a set. For each $\alpha \in I$, consider a set A_α. The set $\{A_\alpha : \alpha \in I\}$ is called a family of sets indexed by elements of I. Here I is called an index set and its elements are called index.

The family of sets $\{A_\alpha : \alpha \in I\}$ is called a non-empty family when the index set I is non-empty. Let $\{Y_\alpha : \alpha \in I\}$ be a family of index set. Then:

- The union is defined as $\bigcup_{\alpha \in I} Y_\alpha = \{y : y \in Y_\alpha \, \text{for some} \, \alpha \in I\}$.
- The intersection is defined as $\bigcap_{\alpha \in I} Y_\alpha = \{y : y \in Y_\alpha \, \text{for all} \, \alpha \in I\}$.
- **Convention:** The union of sets in an empty family is ϕ. The intersection of sets in an empty family of subsets of S is S.

Product of infinite family of sets: Let A_1 and A_2 be two non-empty sets. Then $A_1 \times A_2$ is identified with a set of all functions $f : \{1, 2\} \to A_1 \cup A_2$ with $f(1) \in A_1$ and $f(2) \in A_2$. For example, let $A_1 = \{a, b\}$ and $A_2 = \{c, d\}$. Then $A_1 \times A_2 = \{(a, c), (a, d), (b, c), (b, d)\}$. Define

$f_1 : \{1, 2\} \to A_1 \cup A_2$ such that $f_1(1) = a$, $f_1(2) = c$ gives (a, c); $f_2 : \{1, 2\} \to A_1 \cup A_2$ such that $f_2(1) = b$, $f_2(2) = c$ gives (b, c); $f_3 : \{1, 2\} \to A_1 \cup A_2$ such that $f_3(1) = a$, $f_3(2) = d$ gives (a, d); $f_4 : \{1, 2\} \to A_1 \cup A_2$ such that $f_4(1) = b$, $f_4(2) = d$ gives (b, d).

Thus $A_1 \times A_2 = \{f_1, f_2, f_3, f_4\}$. Generalizing this notion leads to the following:

Let $\{A_\alpha\}_{\alpha \in I}$ be a non-empty family of sets. Assume that A_α is also non-empty for each $\alpha \in I$. Then the product the sets in the family is defined as:

$\prod_{\alpha \in I} A_\alpha = \{f : I \to \cup_{\alpha \in I} A_\alpha \, \text{such that} \, f(\alpha) \in A_\alpha \forall \alpha \in I\}$.